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# SINGULARITIES INTERACTING WITH INTERFACES AND CRACKS

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**Abstract**—Solutions to singularities such as point force, point moment, edge dislocation and transformation strain spot embedded in bonded elastic blocks of dissimilar materials are found to relate to the solutions to the same singularities in an infinite homogeneous plane by a formula independent of the nature of the singularities. This universal result is then used to analyze the interactions between singularities and interface cracks. The complete solutions and stress intensity factors are presented for two important interface crack configurations.

## 1. INTRODUCTION

Recent interest in micromechanics calls for the analyses of the elastic fields for *pointwise singularities*, such as point force, point moment, edge dislocation and circular transformation strain spot, interacting with interfaces and cracks. For example, crack/dislocation interaction plays an important role in understanding the brittle vs ductile response of crystals (Thomson, 1986); enhanced toughness in $ZrO_2$-particle-enriched ceramics has been modeled successfully in recent years, e.g. Budiansky *et al.* (1983), where an analysis of the transformation strain spot interacting with cracks is usually the first step; embrittlement/ductilization of polycrystals by impurity segregation have received much attention recently, mechanistic modelling of which has been attempted by considering crack tip (anti-)shielding by impurities (essentially dilatation spots) in Weertman and Hack (1988).

Along with the intrinsic physical significance of these interaction solutions, they are frequently used as kernel functions of integral equations. A well-known approach to simulate cracks by arrays of dislocations was explored extensively by Erdogan (1972). Recent applications are made by Hutchinson and his group to model phenomena such as crack kinking, edge spalling, composite delamination, and to analyze some interface fracture specimens. As suggested by the success of a method of analyzing homogeneous cracks (Mews and Kuhn, 1988), use of point force solutions in a cracked bimaterial system as the fundamental solutions, in conjunction with a procedure to extract mixed mode stress intensity factors from path-independent integrals (Shih and Asaro, 1988), may lead to an efficient Boundary Element algorithm for interface crack analyses.

Apparently there are many applications for this class of solutions, which may partially justify a unified presentation for such classical-looking problems. Another fact is that most of the work cited above is confined to cracks in homogeneous materials, and consequently, most singularity/crack interaction solutions scattered in the literature are for homogeneous materials. As a matter of fact, this note results from the investigation of the author, and the research group to which he belongs, on the mechanics of thin film and interface fracture (He and Hutchinson, 1988a,b; Suo and Hutchinson, 1988a,b, 1989). Attention here will be focussed on the construction of the basic solutions. Guidelines for sophisticated applications may be found in the above-mentioned papers.

The plane elasticity problem analyzed is depicted in Fig. 1. A singularity interacting with the bimaterial interface is considred first (Fig. 1a). Without loss of generality, singularities are only embedded in material 2. The solution is built on the complex potentials for the same singularity in an infinite homogeneous plane. As illustrated in Fig. 1, the interaction between singularities and interfacial cracks is analyzed by superposition. To make the scheme possible, one needs the solution to the problem specified in Fig. 1b, with interfacial cracks loaded by traction on the crack faces. This latter problem has been solved by several

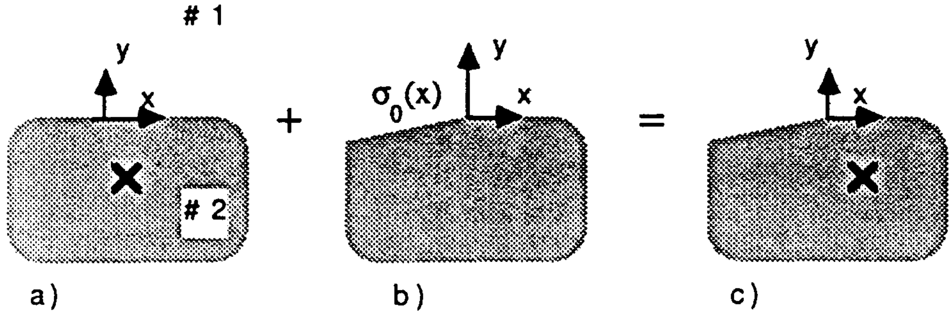


Fig. 1. Superposition scheme. (a) A singularity embedded in one of two well bonded blocks. (b) Interface crack with traction prescribed on the faces. (c) Interaction between a singularity with a traction-free crack.

authors (England, 1965; Erdogan, 1965; Rice and Sih, 1965; Cherepanov, 1979), and will be adapted here by using the two Dundurs' parameters defined below.

The non-dimensional elastic moduli dependence of a bimaterial system, with simply connected domain and traction prescribed on its boundary, may be expressed in terms of two *Dundurs' parameters* (Dundurs, 1968)

$$\alpha = \frac{\Gamma(\kappa_2+1)-(\kappa_1+1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)}, \quad \beta = \frac{\Gamma(\kappa_2-1)-(\kappa_1-1)}{\Gamma(\kappa_2+1)+(\kappa_1+1)}. \tag{1}$$

Subscripts 1 and 2 refer to the two materials, $\kappa = 3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress, $\Gamma = \mu_1/\mu_2$, $\nu$ is Poisson's ratio and $\mu$ is shear modulus. The physically admissible values of $\alpha$ and $\beta$ are restricted to a parallelogram enclosed by $\alpha = \pm 1$ and $\alpha - 4\beta = \pm 1$ in the $\alpha, \beta$-plane. The two parameters measure the elastic dissimilarity of two materials in the sense that both vanish when the dissimilarity does. Two other bimaterial parameters, $\Sigma$, the *stiffness ratio*, and $\varepsilon$, the *oscillatory index*, are related to $\alpha$ and $\beta$, respectively, by

$$\Sigma = \frac{c_2}{c_1} = \frac{1+\alpha}{1-\alpha}, \quad \varepsilon = \frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta} \tag{2}$$

where $c = (\kappa+1)/\mu$ is a measure of the compliance of a material and will appear again. Thus $\alpha$ can be readily interpreted as a measure of the dissimilarity in stiffness of the two materials. Material 1 is stiffer than 2 as $\alpha > 0$ and the material 1 is relatively compliant as $\alpha < 0$. The parameter $\varepsilon$, thus $\beta$, as has been discussed extensively in the literature on interfacial fracture mechanics, is responsible for various pathological behaviors at an interfacial crack tip (e.g. Rice, 1988). However, $\varepsilon$ is typically very small. Indeed, since $|\beta| \leqslant 0.5$, from (2) one finds $|\varepsilon| \leqslant \ln(3)/2\pi \approx 0.175$. Various proposals for handling or ignoring the $\varepsilon$-effects have been considered (Suo and Hutchinson, 1988b; Rice, 1988). In this paper no special consideration is given to such effects as crack face contact due to non-zero $\varepsilon$ in deriving the results for the crack/singularity interaction.

## 2. COMPLEX POTENTIALS

Stresses and displacements for a homogeneous body under plane deformation can be represented by two standard Muskhelishvili complex potentials $\phi(z)$ and $\psi(z)$. However, another pair of commonly used potentials, $\Phi(z)$ and $\Omega(z)$, defined as

$$\Phi(z) = \phi'(z), \quad \Omega(z) = [z\phi'(z)+\psi(z)]', \tag{3}$$

prove to be more convenient for our purpose. Stress displacement components are then derived from

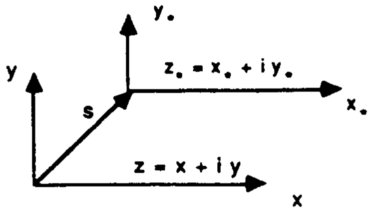


Fig. 2. A coordinate translation.

$$\sigma_{xx}+\sigma_{yy} = 2[\Phi(z)+\overline{\Phi(z)}]$$
$$\sigma_{yy}+i\sigma_{xy} = \overline{\Phi(z)}+\Omega(z)+(\bar{z}-z)\Phi'(z)$$
$$-2i\mu\frac{\partial}{\partial x}(u_y+iu_x) = \kappa\overline{\Phi(z)}-\Omega(z)-(\bar{z}-z)\Phi'(z). \tag{4}$$

One can confirm a useful coordinate translation rule (see Fig. 2). Suppose that $\Phi_*(z_*)$ and $\Omega_*(z_*)$ are the potentials in the coordinate system $z_* = x_*+iy_*$, while $\Phi(z)$ and $\Omega(z)$ are the potentials for the same problem in the coordinate system $z = x+iy$, where $z_* = z-s$, then

$$\Phi(z) = \Phi_*(z-s), \quad \Omega(z) = \Omega_*(z-s)+(s-\bar{s})\Phi'_*(z-s). \tag{5}$$

Potentials for singularities in an *infinite homogeneous plane* are the building blocks of this paper. Listed below are some frequently used examples.

*An edge dislocation at* z = s

$$\Phi_0(z) = B\left[\frac{1}{z-s}\right], \quad \Omega_0(z) = B\left[\frac{\bar{s}-s}{(z-s)^2}\right]+\bar{B}\left[\frac{1}{z-s}\right]$$
$$B = \frac{\mu}{\pi i(1+\kappa)}(b_x+ib_y) \tag{6}$$

where $b_x$ and $b_y$ are the $x$- and $y$-components of the dislocation.

*A point force at* z = s

$$\Phi_0(z) = -Q\left[\frac{1}{z-s}\right], \quad \Omega_0(z) = -Q\left[\frac{\bar{s}-s}{(z-s)^2}\right]+\kappa\bar{Q}\left[\frac{1}{z-s}\right]$$
$$Q = \frac{1}{2\pi(\kappa+1)}(P_x+iP_y) \tag{7}$$

where $P_x$ and $P_y$ are the force components in the $x$ and $y$ directions.

*A point moment $M$ at* z = s

$$\Phi_0(z) = 0, \quad \Omega_0(z) = \frac{M}{2\pi i}\left[\frac{1}{(z-s)^2}\right] \tag{8}$$

*A circular transformation strain spot*

Let a circular region of radius $R$ and center $z = s$ in an infinite homogeneous plane undergo a uniform transformation straining $\varepsilon_{\alpha\beta}$. Continuity of tractions and displacements across the circular boundary is maintained. This is a 2D version of the Eshelby problem, which is included in an unpublished report by Hutchinson (1974). The potentials for the elastic field outside the circular spot, $|z-s| > R$, are

$$\Phi_0(z) = -AR^2\left[\frac{1}{(z-s)^2}\right]$$

$$\Omega_0(z) = (A+4B)R^2\left[\frac{1}{(z-s)^2}\right]-3AR^4\left[\frac{1}{(z-s)^4}\right]+2AR^2\left[\frac{s-\bar{s}}{(z-s)^3}\right]$$

$$A = \frac{\mu}{1+\kappa}(\varepsilon_{xx}-\varepsilon_{yy}+2i\varepsilon_{xy}), \quad B = \frac{\mu}{1+\kappa}\left(\frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}\right). \tag{9}$$

## 3. SINGULARITIES INTERACTING WITH A BIMATERIAL INTERFACE

Now the problem in Fig. 1a is considered. Let the potentials for the two blocks be

$$\Phi(z) = \begin{cases}\Phi^1(z)+\Phi_0(z), & z \text{ in No. 1}\\ \Phi^2(z)+\Phi_0(z), & z \text{ in No. 2}\end{cases} \quad \Omega(z) = \begin{cases}\Omega^1(z)+\Omega_0(z), & z \text{ in No. 1}\\ \Omega^2(z)+\Omega_0(z), & z \text{ in No. 2}\end{cases} \tag{10}$$

where $\Phi_0(z)$ and $\Omega_0(z)$ signify the potentials for a singularity in an *infinite homogeneous plane* of material 2, which could be one of those listed in Section 2. Obviously $\Phi_0(z)$, $\Omega_0(z)$, $\Phi^1(z)$ and $\Omega^1(z)$ are analytic for $z$ above the $x$-axis, while $\Phi^2(z)$ and $\Phi^2(z)$ are analytic for $z$ below the $x$-axis. The task below is to relate $\Phi^1(z)$, $\Omega^1(z)$, $\Phi^2(z)$ and $\Omega^2(z)$ to $\Phi_0(z)$ and $\Omega_0(z)$. The continuity of $\sigma_{yy}+i\sigma_{xy}$ across the interface requires

$$\Phi^1(x)+\Omega^1(x) = \Phi^2(x)+\Omega^2(x). \tag{11}$$

By the standard analytic continuation arguments it follows that

$$\Phi^1(z) = \Omega^2(z), \quad z \text{ in No. 2}$$

$$\Phi^2(z) = \Omega^1(z), \quad z \text{ in No. 1}. \tag{12}$$

The continuity of displacements across the interface, with the aid of (10), leads to

$$(1-\beta)\Omega^2(x)-(\alpha+\beta)\bar{\Phi}_0(x) = (1+\beta)\Omega^1(x)-(\alpha-\beta)\Omega_0(x). \tag{13}$$

Again by analytic continuity arguments one obtains

$$\Omega^2(z) = \Lambda\bar{\Phi}_0(z), \quad \Omega^1(z) = \Pi\Omega_0(z). \tag{14}$$

Here $\Lambda$ and $\Pi$ measure the inhomogeneity by

$$\Lambda = \frac{\alpha+\beta}{1-\beta}, \quad \Pi = \frac{\alpha-\beta}{1+\beta}. \tag{15}$$

Now with (12) and (14) one can rewrite (10) explicitly as

$$\Phi(z) = \begin{cases}(1+\Lambda)\Phi_0(z), & z \text{ in No. 1}\\ \Phi_0(z)+\Pi\bar{\Omega}_0(z), & z \text{ in No. 2}\end{cases} \quad \Omega(z) = \begin{cases}(1+\Pi)\Omega_0(z), & z \text{ in No. 1}\\ \Omega_0(z)+\Lambda\bar{\Phi}_0(z), & z \text{ in No. 2}\end{cases} \tag{16}$$

so that the singularity solutions in bonded half-planes of dissimilar materials can be constructed using the corresponding singularity solutions in an infinite homogeneous plane by eqn (16). This relation is universal in the sense that it is completely independent of the physical nature of the singularities.

Singularities in a half-space interacting with the traction-free surface can be treated as a special case by letting $\alpha = -1$, or $\Lambda = \Pi = -1$. Specializing (16) to this case one obtains

$$\Phi(z) = \Phi_0(z) - \overline{\Omega_0}(z), \quad \Omega(z) = \Omega_0(z) - \overline{\Phi_0}(z). \tag{16a}$$

## 4. INTERFACIAL CRACKS

The singular stress field of an interfacial crack tip shows an $r^{-1/2+i\varepsilon}$ type singularity. Accordingly, the *complex stress intensity factor*, $K = K_1 + iK_2$, is defined (Hutchinson *et al.*, 1987) such that the traction in the interface a distance $r$ ahead of the crack tip is

$$\sigma_{yy} + i\sigma_{xy} = \frac{K}{\sqrt{2\pi r}} r^{i\varepsilon} \tag{17}$$

and the relative crack face displacements a distance $r$ behind the crack tip are given by

$$\delta_y + i\delta_x = \frac{c_1 + c_2}{2\sqrt{2\pi}(1+i\varepsilon)\cosh(\pi\varepsilon)} K\sqrt{r} r^{i\varepsilon}. \tag{18}$$

thereby the energy release rate is

$$G = \frac{c_1 + c_2}{16\cosh^2 \pi\varepsilon} |K|^2. \tag{19}$$

To make the superposition scheme in Fig. 1 possible, one needs the *complete* solution for the problem in Fig. 1b. Suppose the cracks considered lie on the interface of two dissimilar material blocks. It suffices to consider only the case of traction prescribed on the crack faces, for other methods of loading may be reduced to this case by superposition. This problem was solved in 1965 by England, Erdogan, and Rice and Sih. Outlined below is the solution in the present notation. The derivation is simplified to some extent by the use of the Dundurs' parameters.

Let the potentials for the two half-planes in Fig. 1b be written as

$$\Phi(z) = \begin{cases} \Phi^a(z), & z \text{ in No. 1} \\ \Phi^b(z), & z \text{ in No. 2} \end{cases} \quad \Omega(z) = \begin{cases} \Omega^a(z), & z \text{ in No. 1} \\ \Omega^b(z), & z \text{ in No. 2} \end{cases} \tag{20}$$

where the superscript "a" indicates that the potential is for the material above, while "b" is for the material below. The continuity of $\sigma_{yy} + i\sigma_{xy}$ across the interface requires

$$\begin{aligned} \overline{\Phi^a}(z) &= \Omega^b(z), \quad z \text{ in No. 2} \\ \overline{\Phi^b}(z) &= \Omega^a(z), \quad z \text{ in No. 1}. \end{aligned} \tag{21}$$

With (21) one can show that the derivative of displacement jumps across the interface, or the components of the Burgers vector, can be written as

$$-2i\frac{\partial}{\partial x}(\delta_y + i\delta_x) = \frac{c_1 + c_2}{2}[(1-\beta)\Omega^b(x) - (1+\beta)\Omega^a(x)]. \tag{22}$$

Consequently, due to the continuity of the displacement across the bonded portion of the interface, one can define a function $f(z)$ which is analytic in the whole plane except on the crack lines, such that

$$\begin{aligned} \Omega^a(z) &= (1-\beta)f(z), \quad z \text{ in No. 1} \\ \Omega^b(z) &= (1+\beta)f(z), \quad z \text{ in No. 2}. \end{aligned} \tag{23}$$

In terms of $f(z)$, the Burgers vector (22) can be written as

$$-2i\frac{\partial}{\partial x}(\delta_y+i\delta_x) = \frac{c_1+c_2}{2}(1-\beta^2)[f^-(x)-f^+(x)] \tag{24}$$

and the traction on the interface is given by

$$\sigma_{yy}+i\sigma_{xy} = (1+\beta)f^-(x)+(1-\beta)f^+(x). \tag{25}$$

The prescribed traction on the cracks thus leads to the following Hilbert problem

$$(1+\beta)f^-(x)+(1-\beta)f^+(x) = \sigma_{yy}(x)+i\sigma_{xy}(x), \quad \text{on crack lines.} \tag{26}$$

Suppose there are $n$ finite cracks in the intervals $(a_j, b_j)$ and two semi-infinite cracks in the intervals $(-\infty, b_0)$ and $(a_0, +\infty)$, respectively, on the $x$-axis. Following the methods of Muskhelishvili (1953), a homogeneous solution of eqn (26) [i.e. a solution of $f(z)$ when setting the right-hand side of eqn (26) to be zero] can be written as

$$\chi(z) = \prod_{j=0}^{n}(z-a_j)^{-1/2-i\varepsilon}(z-b_j)^{-1/2+i\varepsilon} \tag{27}$$

where the branch cuts are chosen along the crack lines so that the product for each finite crack behaves as $1/z$ for large $z$. The solution to (26) is

$$f(z) = \frac{1}{1-\beta}\frac{\chi(z)}{2\pi i}\int\frac{\sigma_{yy}(x)+i\sigma_{xy}(x)}{\chi^+(x)(x-z)}\,dx+\chi(z)P(z) \tag{28}$$

where the integral should be taken over the union of the cracks, and $P(z)$ is a polynomial which should be chosen so that $f(z)$ is bounded at infinity and the net Burgers vector for each of the $n$ finite cracks is zero. From (24) this latter statement leads to $n$ equations

$$\int_{a_j}^{b_j}[f^-(x)-f^+(x)]\,dx = 0, \quad j = 1, 2, \ldots, n. \tag{29}$$

Thus $f(z)$ can be determined and also the potentials $\Phi(z)$ and $\Omega(z)$ by (20), (21) and (23). Once $f(z)$ is obtained, the complex stress intensity factor defined in (17) can be extracted from (25). Noticing that $f^-(x) = f^+(x) = f(x)$ on the bonded portion of the interface, one obtains

$$K = \sqrt{2\pi}\lim_{x\to a} 2(x-a)^{1/2-i\varepsilon}f(x) \tag{30}$$

if the crack tip is at $x = a$, running in the direction of positive $x$-axis.

Two configurations depicted in Fig. 3 are of particular importance in the application. The cracks are loaded by equal but opposite tractions $\sigma_{yy}+i\sigma_{xy} = -T(x)$ on the crack faces. One can verify for the semi-infinite crack

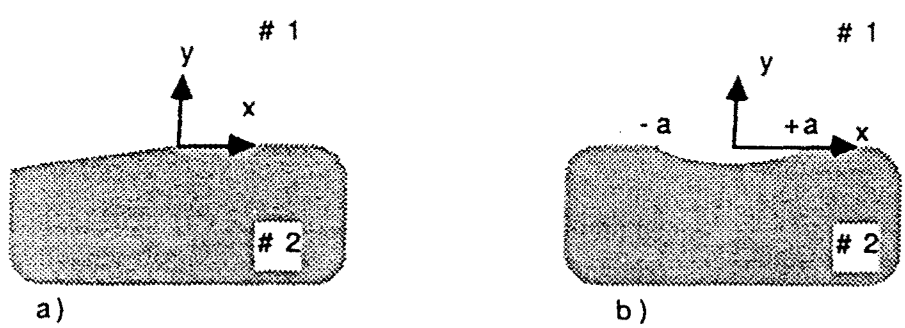


Fig. 3. Two crack configurations.

$$\chi(z) = z^{-1/2+i\varepsilon}, \quad P(z) = 0. \tag{31}$$

Thus $f(z)$ can be determined from (28), and the complex stress intensity factor is given by

$$K = \sqrt{\frac{2}{\pi}} \cosh \pi\varepsilon \int_{-\infty}^{0} \frac{T(t)}{(-t)^{1/2+i\varepsilon}} \, dt. \tag{32}$$

In the case of a finite crack of length $2a$ the corresponding results are

$$\chi(z) = (z-a)^{-1/2+i\varepsilon}(z+a)^{-1/2-i\varepsilon}, \quad P(z) = 0, \tag{33}$$

and the stress intensity factor at the right-hand side tip is

$$K = \sqrt{\frac{2}{\pi}} \cosh \pi\varepsilon (2a)^{-1/2-i\varepsilon} \int_{-a}^{+a} \left(\frac{a+t}{a-t}\right)^{1/2+i\varepsilon} T(t) \, dt. \tag{34}$$

A short list of stress intensity factors for some special loading cases can be found in Shih and Asaro (1988).

## 5. SINGULARITIES INTERACTING WITH INTERFACIAL CRACKS

Now the interaction problem illustrated in Fig. 1c can be readily solved by the superposition of the solutions obtained in the last two sections. This scheme was used by Thomson (1986) to construct the interaction solution of a dislocation and a crack in a homogeneous body.

The potentials for a singularity at $z = s$ in an infinite homogeneous plane can be written in general as

$$\Phi_0(z) = \sum_{m=1}^{M} \frac{A_m}{(z-s)^m}, \quad \Omega_0(z) = \sum_{m=1}^{M} \frac{B_m}{(z-s)^m} \tag{35}$$

where the coefficients $A_m$ and $B_m$ may depend on $s$ and the nature of the singularity. Several examples are given in Section 2. Suppose the potentials (35) are known, the potentials for the same singularity embedded in one of two bonded blocks as in Fig. 1a are readily constructed by (16). In particular, the stresses along the interface in Fig. 1a are

$$\sigma_{yy}(x) + i\sigma_{xy}(x) = (1+\Lambda)\overline{\Phi_0}(x) + (1+\Pi)\Omega_0(x). \tag{36}$$

The negatives of these stresses are applied to the faces of the crack in Fig. 1b. It follows from (28) that

$$f(z) = -\frac{1}{1-\beta}\frac{\chi(z)}{2\pi i}\int \frac{(1+\Lambda)\overline{\Phi_0}(x) + (1+\Pi)\Omega_0(x)}{\chi^+(x)(x-z)} \, dx. \tag{37}$$

Here we have set $P(z) = 0$. One has to specify the crack configuration in order to evaluate the integral

$$I = \frac{1}{2\pi i}\int \frac{(1+\Lambda)\overline{\Phi_0}(x) + (1+\Pi)\Omega_0(x)}{\chi^+(x)(x-z)} \, dx \tag{38}$$

where the integral should be taken on the crack lines. Only the two configurations depicted in Fig. 3 will be analyzed below. Consider the following contour integral

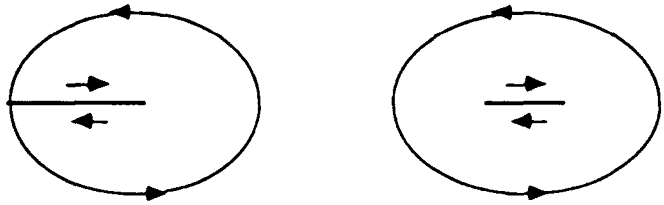

Fig. 4. Integration contours.

$$J = \frac{1}{2\pi i}\oint \frac{(1+\Lambda)\Phi_0(\xi)+(1+\Pi)\Omega_0(\xi)}{\chi(\xi)(\xi-z)}\,\mathrm{d}\xi \tag{39}$$

with the contours specified in Fig. 4. It is easy to confirm

$$J = J_\infty + \frac{2}{1-\beta} I \tag{40}$$

where $J_\infty$ is the same integral as (39) integrated over a circle $|z| = R$ as $R \to \infty$. It can be shown that

$$J_\infty = (1+\Lambda)\bar{A}_1 + (1+\Pi)B_1 \tag{41}$$

for the finite crack and $J_\infty = 0$ for the semi-infinite crack. On the other hand $J$ can be evaluated by its residues

$$J = \frac{1}{\chi(z)}[(1+\Lambda)\Phi_0(z)+(1+\Pi)\Omega_0(z)] + \sum_{m=1}^{M}[(1+\Lambda)\bar{A}_m F_{m-1}(z,\bar{s}) + (1+\Pi)B_m F_{m-1}(z,s)] \tag{42}$$

where

$$F_m(z,s) = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}s^m}\left[\frac{1}{\chi(s)(s-z)}\right]. \tag{43}$$

Consequently, one obtains

$$f(z) = -\tfrac{1}{2}[(1+\Lambda)\overline{\Phi_0}(z)+(1+\Pi)\Omega_0(z)] - \frac{\chi(z)}{2}\sum_{m=1}^{M}[(1+\Lambda)\bar{A}_m F_{m-1}(z,\bar{s}) + (1+\Pi)B_m F_{m-1}(z,s)] \tag{44}$$

for the semi-infinite crack and

$$f(z) = \frac{\chi(z)}{2}[(1+\Lambda)\bar{A}_1 + (1+\Pi)B_1] - \tfrac{1}{2}[(1+\Lambda)\bar{\Phi}_0(z)+(1+\Pi)\Omega_0(z)] - \frac{\chi(z)}{2}\sum_{m=1}^{M}[(1+\Lambda)\bar{A}_m F_{m-1}(z,\bar{s}) + (1+\Pi)B_m F_{m-1}(z,s)] \tag{45}$$

for the finite crack. Therefore the potentials for the problem in Fig. 1b can be obtained from (20), (21) and (23) with $f(z)$ given in (44) and (45). The potentials for the interaction problem in Fig. 1c are thus the superposition of (16) and those for Fig. 1b.

Since no stress intensity is present in the structure in Fig. 1a, one can deduce the stress intensity factors from (44) and (45). The results are

$$K = -\sqrt{2\pi} \sum_{m=1}^{M} [(1+\Lambda)\bar{A}_m F_{m-1}(0,\bar{s}) + (1+\Pi)B_m F_{m-1}(0,s)]$$

$$F_m(0,s) = \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}s^m}[s^{-1/2-i\varepsilon}] \tag{46}$$

for the semi-infinite crack and

$$K = +\sqrt{2\pi}(2a)^{-1/2-i\varepsilon}[(1+\Lambda)\bar{A}_1 + (1+\Pi)B_1]$$
$$-\sqrt{2\pi}(2a)^{-1/2-i\varepsilon} \sum_{m=1}^{M} [(1+\Lambda)\bar{A}_m F_{m-1}(a,\bar{s}) + (1+\Pi)B_m F_{m-1}(a,s)]$$

$$F_m(a,s) = \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}s^m}\left[\left(\frac{s+a}{s-a}\right)^{1/2+i\varepsilon}\right] \tag{47}$$

for the finite crack.

## 6. SUMMARY

Complete solutions to three problems for bimaterial systems specified in Fig. 1 can be found in this paper. Singularities of an arbitrary physical nature are treated within the same framework. In particular, a universal relation is found between the potentials for singularities in an infinite homogeneous plane and in bonded blocks of dissimilar materials [eqn (16) above]. Stress intensity factors are identified for the two crack configurations for both face-loaded problem [see Fig. 1b and eqns (32, 34)] and the interaction problem (see Fig. 1c and eqns (46, 47)]. It is believed that the present work will be helpful for those who are interested in micromechanics modelling as illustrated in the Introduction. Parallel results for cracks on the interface between dissimilar anisotropic bodies will be reported in the sequel of this paper (Suo, 1989).

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